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Algorithm for a posteriori error bounds of the numerical solution for initial value problems by discrete variable methods (I)

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1. Introduction

In the paper [2], We considered an initial value problem

$$\frac{dx}{dt} = X(x,t) \tag{1.1}$$

with an initial condition

$$x(a) = \ell, \tag{1.2}$$

where x, $X(x,t)$ and ℓ are real vectors of the same dimension. We have then obtained a global existence theorem of an exact solution in a neighborhood of a continuous approximate solution and proposed an algorithm for getting a global error bound of a discrete numerical solution over an interval $I=[a,b]$, when the above differential system is integrated numerically by means of a discrete variable method over I.

Several global existence theorems and their applications to getting global error bounds have been discussed in M. Urabe

[6], H. Shintani and Y. Hayashi [5], Y. Hayashi [4] and T. Yamamoto [7] as the special case of boundary value problems including an initial value problem. But any algorithm to the case of the discrete numerical solution has not been given in them. For two-point boundary value problems, G. Kedem [1] has proposed the algorithm in which a method of the interval analysis is used. But he has dealt with only the case that $X(x,t)$ is a polynomial in t and x.

In this paper, we propose a new algorithm which is more accurate than that given in [2]. We assume that the dimension of the differential equation in (1.1) is one, the order of the discrete variable method is four and the step-size is a constant h. But in Section 2 we shall describe the global existence theorem in vector forms and in Section 3 also mention continuous approximate solutions in vector forms. In Section 4 we shall give quadrature formulas which are important in our algorithm, in Section 5 state an algorithm for evaluating some necessary quantities for our purpose and in Section 6 discuss the accuracies of them. In Section 7 a numerical example will be shown and in Appendix some remarks will be given.

Computations in this paper have been carried out by the use of FACOM M-382 at Kyushu University and some preparatory computations by the use of FACOM S-3300 at Fukuoka University of Education.

2. The global existence theorem

By the symbol $\|\cdot\|$, we denote a suitable norm of a vector or a matrix. For a vector-or a matrix-valued function defined on I, we denote the supremum (over I) of the above norm by $\|\cdot\|_c$.

Let D' be a domain of the tx-space intercepted by two hyperplanes $t=a$ and $t=b$ such that the cross sections R_a and R_b at $t=a$ and $t=b$ make an open set in each hyperplane, and $D=R_a \cup D' \cup R_b$ on which $X(x,t)$ is defined.

Let us denote the knots on I by t_n $(n=0,1,\ldots,N)$, where

$$a = t_0 < t_1 < \ldots < t_n = b. \tag{2.1}$$

For each n, by $x_{(n)}$ we denote the value of the numerical solution of (1.1) and (1.2) at the knot t_n.

Let $x_0(t)$ be a continuous interpolating function passing through the points $(t_n, x_{(n)}) \in D$ $(n=0,1,\ldots,N)$. We shall call such a function $x_0(t)$ a continuous approximate solution, if it lies in the interior of D. Let us define the residual error $r(t)$ as follows:

$$r(t) = x_0(t) - x_{(0)} - \int_a^t X[x_0(s),s]ds \tag{2.2}$$

Let $A(t)$ be a continuous matrix-valued function defined on I and $\Phi(t)$ be the fundamental matrix of the linear homogeneous differential system

$$\frac{dy}{dt} = A(t)y \tag{2.3}$$

satisfying the initial condition

$$\Phi(a) = E \text{ (the unit matrix).} \tag{2.4}$$

Then we have the following

THEOREM 1. Let $X(x,t)$ in (1.1) be continuous and continuously differentiable with respect to x in D. For a continuous approximate solution $x_0(t)$, we suppose that there are a continuous matrix-valued function $A(t)$ defined on I, a positive constant δ and a non-negative constant $\kappa < 1$ such that

$$D_\delta = \{(t,x) \mid \|x - x_0(t)\| \leqq \delta,\ t \in I\} \subset D, \tag{2.5}$$

$$\|X_x(x,t) - A(t)\| \leqq \frac{\kappa}{M_1} \quad \text{for any} \quad (t,x) \in D_\delta, \tag{2.6}$$

(2.7) $$\frac{M_2}{1 - \kappa} \leqq \delta,$$

where $X_x(x,t)$ is the Jacobian matrix of $X(x,t)$ with respect to x, and where M_1 and M_2 are the quantities such that

(2.8) $$M_1 \geqq \max_{t \in I} \int_a^t \| \Phi(t)\Phi^{-1}(s) \| \, ds,$$

(2.9) $$M_2 \geqq \left\| \int_a^t \Phi(t)\Phi^{-1}(s)A(s)r(s)ds + r(t) + \Phi(t)(x_{(0)} - \ell) \right\|_c .$$

Then there exists one and only one solution $\hat{x}(t)$ to the given differental system (1.1) satisfying (1.2) over I. Furthermore, for $x = \hat{x}(t)$ it holds that

(2.10) $$\| \hat{x} - x_0 \|_c \leqq \frac{M_2}{1 - \kappa} \leqq \delta.$$

For the proof, refer to [2] and [5].

REMARK 1. In general we may take $X_x[x_0(t),t]$ as $A(t)$.

REMARK 2. In this paper, as (2.1) we may consider only the case that

(2.11) $$t_n = a + nh \ (n=0,1,\ldots,N), \ h = \frac{b - a}{N} .$$

3. Continuous approximate solutions

Evidently there may exist infinitely many continuous approximate solutions. In [1], G. Kedem has given a concrete form of $x_0(t)$. However, we do not give any concrete form of $x_0(t)$. We use only the discrete information. We imagine that a suitable one may be chosen by the given differential system and the algorithm given in Section 5.

However, we have the following

THEOREM 2. If $\hat{x}(t)$ is the unieque solution of (1.1) and (1.2) over I. then there exists a continuous approximate solution $x_0(t)$ such that

$$\| \hat{x} - x_0 \|_C = \max_{0 \leqq n \leqq N} \| \hat{x}(t_n) - x_{(n)} \|. \tag{3.1}$$

This theorem is proved by constructing such $x_0(t)$ conretely. Let $x_0(t)$ be the function defined on I by

$$x_0(t) = \hat{x}(t) + \{x_{(n)} - \hat{x}(t_n)\}_m \phi_{n0}(t) \tag{3.2}$$

$$+ \{x_{(n+1)} - \hat{x}(t_{n+1})\}_m \psi_{n0}(t) \quad \text{on each } I_n,$$

where $I_n=[t_n,t_{n+1}]$ $(n=0,1,\ldots,N-1)$ and

$$(3.3)\qquad {}_m\psi_{n0}(t) = \left[\int_{t_n}^{t_{n+1}} \left(1 - \frac{s - t_n}{h}\right)^{m-1} \left(\frac{s - t_n}{h}\right)^{m-1} ds\right]^{-1} \times \int_{t_n}^{t} \left(1 - \frac{s - t_n}{h}\right)^{m-1} \left(\frac{s - t_n}{h}\right)^{m-1} ds \quad \text{on each } I_n,$$

$$(3.4)\qquad {}_m\phi_{n0}(t) = 1 - {}_m\psi_{n0}(t) \quad \text{on each } I_n \quad \text{for } m \geqq 1.$$

Then this $x_0(t)$ satisfies the relation (3.1) in Theorem 2 (cf.[3]). Note that there exist other such ones.

By the assumption that the order of discrete variable methods is four, we may write

$$(3.5)\qquad \hat{x}(t_n) - x_{(n)} = O(h^4) \qquad (n=0,1,\ldots,N).$$

Furthermore, if $x_0(t)$ satisfies the relation (3.1) in Theorem 2, then we may also write

$$(3.6)\qquad \hat{x}(t) - x_0(t) = O(h^4) \qquad \text{for all } t \in I.$$

A numerical experiment shows that there exists much better $x_0(t)$ than that defined in (3.2) (cf.Appendix).

However, since we do not give the concrete form of $x_0(t)$, we can not know whether our $x_0(t)$ satisfies the relation (3.6) or not.

Further discussion about continuous approximate solutions is given in Appendix.

4. Quadrature formulas

In our algorithm for getting global error bounds, the following formulas are important:

$$(4.1) \qquad \int_{(i-1)h}^{ih} f(s)ds = h\sum_{j=0}^{k} {}_k a_{ij} f(jh)$$

$$+ {}_k c_i h^{k+3} f^{(k+1)}(\xi_i), \quad 0 < \xi_i < kh,$$

$$(k=6,7;i=1,2,\ldots,k).$$

As is seen from Table 4, for $k=7$ ${}_7c_4$ is the smallest one in magnitude among ${}_7c_i$ $(i=1,2,\ldots,7)$ and from Table 2, for $k=6$ ${}_6c_3$ and ${}_6c_4$ are both the smallest ones in magnitude among ${}_6c_i$ $(i=1,2,\ldots,6)$. Thus we see that it is desirable to use the fourth formula for $k=7$ and the third or the fourth formula for $k=6$ as often as possible respectively.

Table 1

Coefficients ${}_6a_{ij}$ $\qquad g=60480$

i	$g\cdot{}_6a_{i0}$	$g\cdot{}_6a_{i1}$	$g\cdot{}_6a_{i2}$	$g\cdot{}_6a_{i3}$	$g\cdot{}_6a_{i4}$	$g\cdot{}_6a_{i5}$	$g\cdot{}_6a_{i6}$
1	19087	65112	-46461	37507	-20211	6312	-863
2	-863	25128	46989	-16256	7299	-2088	271
3	271	-2760	30819	37504	-6771	1608	-191
4	-191	1608	-6771	37504	30819	-2760	271
5	271	-2088	7299	-16256	46989	25128	-863
6	-863	6312	-20211	37504	-46461	65112	19087

Table 2

Coefficients $_6c_i$ $e=120960$

$e\cdot{}_6c_1$	$e\cdot{}_6c_2$	$e\cdot{}_6c_3$	$e\cdot{}_6c_4$	$e\cdot{}_6c_5$	$e\cdot{}_6c_6$
1375	-351	191	-191	351	-1375

Table 3

Coefficients $_7a_{ij}$ $e=120960$

i	$e\cdot{}_7a_{i0}$	$e\cdot{}_7a_{i1}$	$e\cdot{}_7a_{i2}$	$e\cdot{}_7a_{i3}$	$e\cdot{}_7a_{i4}$	$e\cdot{}_7a_{i5}$	$e\cdot{}_7a_{i6}$	$e\cdot{}_7a_{i7}$
1	36799	139849	-121797	123133	-88547	41499	-11351	1375
2	-1375	47799	101349	-44797	26883	-11547	2999	-351
3	351	-4183	57627	81693	-20227	7227	-1719	191
4	-191	1879	-9531	68323	68323	-9531	1879	-191
5	191	-1719	7227	-20227	81693	57627	-4183	351
6	-351	2999	-11547	26883	-44797	101349	47799	-1375
7	1375	-11351	41499	-88547	123133	-121797	139849	36799

Table 4

Coefficients $_7c_i$ $f=362800$

$f\cdot{}_7c_1$	$f\cdot{}_7c_2$	$f\cdot{}_7c_3$	$f\cdot{}_7c_4$	$f\cdot{}_7c_5$	$f\cdot{}_7c_6$	$f\cdot{}_7c_7$
-33953	7297	-3233	1497	-3233	7297	-33953

5. An algorithm for evaluating $\hat{M}_1$ and $\hat{M}_2$

Let $\hat{M}_1$ and $\hat{M}_2$ be the quantities of the right hand sides of (2.8) and (2.9), and $\overline{M}_1$ and $\overline{M}_2$ be their approximations respectively.

From the definitions of $r(t)$ and $\Phi(t)$, it holds that

$$(5.1)\qquad r(t_n) = x_{(n)} - x_{(0)} - \int_a^{t_n} X[x_0(s),s]ds$$

$$= x_{(n)} - x_{(n-1)} + r(t_{n-1})$$

$$- \int_{t_{n-1}}^{t_n} X[x_0(s),s]ds$$

and

$$(5.2)\qquad \Phi(t_n) = \exp\{\int_a^{t_n} X_x[x_0(s),s]ds\}$$

$$= \Phi(t_{n-1})\exp\{\int_{t_{n-1}}^{t_n} X_x[x_0(s),s]ds\}.$$

Let r_n and Φ_n be the approximations of $r(t_n)$ and $\Phi(t_n)$ defined respectively in the following

ALGORITHM. $(k=6)$

$r_0 = 0,\ \Phi_0 = 1$

for $n=i=1,2,$ do:

$$r_n = x_{(n)} - x_{(n-1)} + r_{n-1} - h\sum_{j=0}^{6} {}_6a_{ij}X[x_{(j)}, t_j]$$

$$d_n = h\sum_{j=0}^{6} {}_6a_{ij}X_x[x_{(j)}, t_j]$$

$$\Phi_n = \Phi_{n-1}\exp(d_n)$$

for $n=3,4,\ldots, N-3$, do:

$$r_n = x_{(n)} - x_{(n-1)} + r_{n-1} - h\sum_{j=0}^{6} {}_6a_{3j}X[x_{(n+j-3)}, t_{n+j-3}]$$

$$d_n = h\sum_{j=0}^{6} {}_6a_{3j}X_x[x_{(n+j-3)}, t_{n+j-3}]$$

$$\Phi_n = \Phi_{n-1}\exp(d_n)$$

for $i=4,5,6$, do:

$$n = N + i - 6$$

$$r_n = x_{(n)} - x_{(n-1)} + r_{n-1} - h\sum_{j=0}^{6} {}_6a_{ij}X[x_{(n+j-6)}, t_{n+j-6}]$$

$$d_n = h\sum_{j=0}^{6} {}_6a_{ij}X_x[x_{(n+j-6)}, t_{n+j-6}]$$

$$\Phi_n = \Phi_{n-1}\exp(d_n)$$

//

$\overline{M}_2 = 0, \; E = 0$

for $n=i=1,2,$ do :

$$E = E + h\sum_{j=0}^{6} {}_6a_{ij}\Phi_j^{-1}X_x[x_{(j)}, t_j]r_j$$

$$F = |\Phi_n E + r_n + \Phi_n \cdot (x_{(0)} - \ell)|$$

$$\overline{M}_2 = \max(F, \overline{M}_2)$$

for $n=3,4,\ldots,N-3,$ do:

$$E = E + h\sum_{j=0}^{6} {}_6a_{3j}\Phi_{n+j-3}X_x[x_{(n+j-3)}, t_{n+j-3}]r_{n+j-3}$$

$$F = |\Phi_n E + r_n + \Phi_n \cdot (x_{(0)} - \ell)|$$

$$\overline{M}_2 = \max(F, \overline{M}_2)$$

for $i=4,5,6,$ do:

$$n = N + i - 6$$

$$E = E + h\sum_{j=0}^{6} {}_6a_{ij} \;{}_{n+j-6}X_x[x_{(n+j-6)}, t_{n+j-6}]r_{n+j-6}$$

$$F = |\Phi_n E + r_n + \Phi_n \cdot (x_{(0)} - \ell)|$$

$$\overline{M}_2 = \max(F, \overline{M}_2)$$

$$\overline{M}_1 = 0, \quad C = 0$$

for $n=i=1,2,$ do:

$$C = C + h \sum_{j=0}^{6} {}_6a_{ij} \Phi_j^{-1}$$

$$D = |\Phi_n C|$$

$$\overline{M}_1 = \max(D, \overline{M}_1)$$

for $n=3,4,\ldots, N-3,$ do:

$$C = C + h \sum_{j=0}^{6} {}_6a_{3j} \Phi_{n+j-3}^{-1}$$

$$D = |\Phi_n C|$$

$$\overline{M}_1 = \max(D, \overline{M}_1)$$

for $i=4,5,6$, do:

$n = N + i - 6$

$C = C + h \sum_{j=0}^{6} {}_6a_{ij} \Phi^{-1}_{n+j-6}$

$D = |\Phi_n C|$

$\overline{M}_1 = \max(D, \overline{M}_1)$

We have a similar algorithm for $k=7$.

6. The accuracies of r_n, Φ_n, $\overline{M}_1$ and $\overline{M}_2$

For $k=6$, since $r_0 = r(t_0) = 0$, from (4.1), (5.1) and the definition of r_n, we have

$$(6.1) \qquad r_n - r(t_n) = r_{n-1} - r(t_{n-1}) + O(h^8)$$
$$= O(h^7).$$

While since

$$(6.2) \qquad \hat{x}(t_n) - \ell - \int_a^{t_n} X[\hat{x}(s),s]ds = 0,$$

we obtain

$$(6.3) \qquad r(t_n) = x_{(n)} - \hat{x}(t_n) - (x_{(0)} - \ell)$$

$$- \int_a^{t_n} X_x[\hat{x}(s) + \theta(x_0(s) - \hat{x}(s))][x_0(s) - \hat{x}(s)]ds$$

$$0 < \theta < 1.$$

Since $x_{(0)} - \ell$ is negligible, if we suppose that our $x_0(t)$ imagined in Section 3 satisfies the relation (3.6), then it follows that

$$(6.4) \qquad r(t_n) = O(h^4).$$

From (6.1) and (6.3), the relative error of r_n is expressed as follows:

$$(6.5) \qquad \frac{r_n - r(t_n)}{r(t_n)} = O(h^3).$$

From (6.5) we may suppose that r_n has a few significant digits.

From the definitions of d_j and Φ_n, since

$$d_j = \int_{t_{j-1}}^{t_j} X_x[x_0(s), s]ds + O(h^8),$$

we have

$$(6.6) \qquad \Phi_n = \Phi_0 \exp(\sum_{j=1}^{n} d_j)$$

$$= \Phi(t_n)\exp[O(h^7)].$$

Thus the relative error of Φ_n is expressed as follows:

$$(6.7) \qquad \frac{\Phi_n - \Phi(t_n)}{\Phi(t_n)} = \exp[O(h^7)] - 1$$

$$= O(h^7).$$

As is readily seen, we have

$$(6.8) \qquad \frac{\overline{M}_1 - \hat{M}_1}{\hat{M}_1} = O(h^7),$$

$$(6.9) \qquad \frac{\overline{M}_2 - \hat{M}_2}{\hat{M}_2} = O(h^3).$$

Simiraly for k=7, we have

$$(6.10) \qquad \frac{r_n - r(t_n)}{r(t_n)} = O(h^4),$$

$$(6.11) \qquad \frac{\Phi_n - \Phi(t_n)}{\Phi(t_n)} = O(h^8),$$

$$(6.12) \qquad \frac{\overline{M}_1 - \hat{M}_1}{\hat{M}_1} = O(h^8),$$

$$(6.13) \qquad \frac{\overline{M}_2 - \hat{M}_2}{\hat{M}_2} = O(h^4).$$

We denote the approximations $\overline{M}_1$ and $\overline{M}_2$ for the case of k=7 by ${}_7\overline{M}_1$ and ${}_7\overline{M}_2$ respectively, and for the case of k=6 by ${}_6\overline{M}_1$ and ${}_6\overline{M}_2$ respectively.

We do not know the values of $\hat{M}_1$ and $\hat{M}_2$, and so we use ${}_7\overline{M}_1$ and ${}_7\overline{M}_2$ instead of them, respectively in Section 7.

Since there exist constants C_6 and C_7 such that

$$(6.14)\qquad \frac{{}_6\overline{M}_1 - \hat{M}_1}{\hat{M}_1} = C_6 h^7 \quad\text{and}\quad \frac{{}_7\overline{M}_1 - \hat{M}_1}{\hat{M}_1} = C_7 h^8,$$

we see that

$$(6.15)\qquad \frac{{}_6\overline{M}_1 - {}_7\overline{M}_1}{{}_7\overline{M}_1} = C_6 h^7 - C_7 h^8 \cdot \frac{1 + C_6 h^7}{1 + C_7 h^8}$$
$$= \frac{{}_6\overline{M}_1 - \hat{M}_1}{\hat{M}_1} \cdot (1 + O(h)).$$

Simiraly for ${}_6\overline{M}_2$, we have

$$(6.16)\qquad \frac{{}_6\overline{M}_2 - {}_7\overline{M}_2}{{}_7\overline{M}_2} = \frac{{}_6\overline{M}_2 - \hat{M}_2}{\hat{M}_2} \cdot (1 + O(h)).$$

From (6.15) and (6.16), it follows that

$$(6.17)\qquad -\log_{10}\left|\frac{{}_6\overline{M}_i - {}_7\overline{M}_i}{{}_7\overline{M}_i}\right| = -\log_{10}\left|\frac{{}_6\overline{M}_i - \hat{M}_i}{\hat{M}_i}\right| + O(h) \quad (i=1,2).$$

7. A numerical example

Here, as the numerical integration formula, we use the Runge-Kutta-Gill formula with the constant step-size $h=0.01$ in double precision and we compute ${}_k\overline{M}_i$ $(k=6,7;i=1,2)$ in quadruple precision.

EXAMPLE. Consider the initial value problem

(7.1) $$\frac{dx}{dt} = -x^2(2e^t - 1), \quad x(0) = 1, \quad I=[0,1].$$

The exact solution of this problem is given by

(7.2) $$\hat{x}(t) = 1/(2e^t - t - 1).$$

The left hand side of (2.6) reads as follows:

(7.3) $$|X_x(x,t) - X_x[x_0(t),t]| = |4e^t - 2||x - x_0(t)|.$$

At t=0.5, we obtain

$$_6\overline{M}_1 = 0.28809862\ldots, \qquad _6\overline{M}_2 = 0.3254311\ldots\times 10^{-9},$$
$$_7\overline{M}_1 = 0.28809862\ldots, \qquad _7\overline{M}_2 = 0.3254241\ldots\times 10^{-9}.$$

As the approximate significant digits of $_6\overline{M}_2$, we have

$$-\log_{10}\left|\frac{_6\overline{M}_2 - {}_7\overline{M}_2}{_7\overline{M}_2}\right| = 4.666\ldots\ .$$

Thus by rounding up the mantissas of $_6\overline{M}_1$ and $_6\overline{M}_2$ to four decimal digits, we obtain

(7.4) $$M_1 = 0.2881 \quad \text{amd} \quad M_2 = 0.3255\times 10^{-9}$$

respectively.

Since $4e^{0.5} - 2 = 4.595$, from (2.6) and (2.7) we obtain the following inequalities:

$$(7.5) \qquad \frac{M_2}{1 - \kappa} \leqq \delta \leqq \frac{\kappa}{4.595 M_1} .$$

If we take $\kappa = 0.0001$, then we have

$$(7.6) \qquad 0.3256\times10^{-9} \leqq \delta \leqq 0.7553\times10^{-4}.$$

The global error reads as follows:

$$(7.7) \qquad \max_{0\leqq n\leqq 50} |\hat{x}(t_n) - x_{(n)}| = 0.325424117\ldots\times10^{-9}.$$

Hence we see that the global error bound 0.3256×10^{-9} is very good.

At $t=0.8$, the global error takes its maximum in maguni-tude on I. We obtain

$$_6\overline{M}_1 = 0.3407682\ldots , \qquad _6\overline{M}_2 = 0.39406942\ldots\times10^{-9},$$
$$_7\overline{M}_1 = 0.3407682\ldots , \qquad _6\overline{M}_2 = 0.39406473\ldots\times10^{-9},$$

$$- \log_{10}\left|\frac{_6\overline{M}_2 - {_7\overline{M}_2}}{_7\overline{M}_2}\right| = 4.929\ldots .$$

In the same way as in the above case, we have

(7.8) $$M_1 = 0.3408 \quad \text{and} \quad M_2 = 0.3941\times10^{-9}$$

respectively.

Since $4e^{0.8}-2 = 6.903$, if we take $\kappa = 0.0001$, then we have

(7.9) $$0.3942\times10^{-9} \leqq \delta \leqq 0.1448\times10^{-4}.$$

The global error reads as follows:

(7.10) $$\max_{0\leqq n\leqq 100} |\hat{x}(t_n) - x_{(n)}| = 0.39406487\ldots\times10^{-9}.$$

Hence we see that the result is very good.

Appendix

The polynomials ${}_m\phi_{n0}(t)$ and ${}_m\psi_{n0}(t)$ of degree 2m-1 given in (3.3) and (3.4) satisfy the following relations respectively:

(A.1) $$\int_{t_n}^{t_{n+1}} {}_m\phi_{n0}(s)ds = \int_{t_n}^{t_{n+1}} {}_m\psi_{n0}(s)ds$$

$$= \frac{h}{2} \qquad \text{for all} \quad m \geqq 1.$$

For instance, for the initial value problem

$$\frac{dx}{dt} = x, \quad x(0) = 1, \quad I = [0.1], \tag{A.2}$$

$r(t_{n+1})$ corresponding to the $x_0(t)$ given in (3.2) is obtained succesively by

$$r(t_{n+1}) = r(t_n) - \{\hat{x}_{(n)} - x(t_n)\}(1 + \frac{h}{2}) \tag{A.3}$$

$$+ \{x_{(n+1)} - \hat{x}(t_{n+1})\}(1 - \frac{h}{2}) \quad \text{on each } I_n.$$

Let us also consider $x_0(t)$ defined by

$$x_0(t) = \hat{x}(t) + \{x_{(n)} - \hat{x}(t_n)\}{}_2\phi_{n0}(t) \tag{A.4}$$

$$+ \{x_{(n+1)} - \hat{x}(t_{n+1})\}{}_2\psi_{n0}(t)$$

$$+ \{X(x_{(n)},t_n) - X[\hat{x}(t_n),t_n]\}{}_2\phi_{n1}(t)$$

$$+ \{X(x_{(n+1)},t_{n+1}) - X[\hat{x}(t_{n+1}),t_{N=1}]\}{}_2\psi_{n1}(t)$$
$$\text{on each } I_n,$$

where

$${}_2\phi_{n1}(t) = h(1 - \frac{t - t_n}{h})^2(\frac{t - t_n}{h}), \tag{A.5}$$

$$(A.6) \qquad {}_2\psi_{n1}(t) = h\left(\frac{t - t_n}{h}\right)^2\left(\frac{t - t_n}{h} - 1\right).$$

This $x_0(t)$ does not satisfy the relation (3.1) but it satisfies the relation (3.6) and $r(t_{n+1})$ for the initial value problem (A.2) is given by

$$(A.7) \qquad r(t_{n+1}) = r(t_n) - \{x_{(n)} - \hat{x}(t_n)\}\left(1 + \frac{h}{2} + \frac{h^2}{12}\right)$$

$$+ \{x_{(n+1)} - \hat{x}(t_{n+1})\}\left(1 - \frac{h}{2} + \frac{h^2}{12}\right).$$

The numerical results for $h=0.01$ reads as follows:

k	$x_0(t)$	$\overline{M}_2$
7	(3.2)	$0.22463932...\times 10^{-9}$
7	(A.4)	$0.224641326...\times 10^{-9}$
6	Algorithm	$0.2246439629...\times 10^{-9}$
7	Algorithm	$0.2246440034...\times 10^{-9}$

$$|x_{(100)} - \hat{x}(t_{100})| = 0.22464400363...\times 10^{-9}$$

From the above results, we see that our algorithm gives the good results.

References

[1] G. Kedem, A posteriori error bounds for two-point boundary value problems, SIAM J. Numer. Anal. Vol. 18 No. 3 (1981), 431-448.

[2] M. Fujii, An a posteriori error estimation of the numerical solution by step-by-step methods for systems of ordinary differential equations, Bull. Fukuoka Univ. Ed. III Vol. 23 (1973), 35-44.

[3] M. Fujii, Some properties of continuous approximate solutions by discrete variable methods, Bull. Fukuoka Unuv. Ed. III Vol. 30 (1981), 29-35.

[4] Y. Hayashi, On a posteriori error estimation in the numerical solution of systems of ordinary differential equations, Hiroshima Math. J. Vol. 9 (1979), 201-243.

[5] H. Shintani and Y. Hayashi, A posteriori error estimates and iteretive methods in the numerical solution of systems of ordinary differential equations, Hiroshima Math. J. Vol. 8 (1978), 101-121.

[6] M. Urabe, An existence theorem for muti-point boundary value problems, Functional Ekvac. Vol. 9 (1966), 43-60.

[7] T. Yamamoto, An existence theorem of boundary value problems and its application to error estimates, Math. Japonica Vol. 27 (1982), 301-318.